new mixing length formula for power-law fluids has been proposed. Though the solutions have been obtained from $t = 0$ to the steady state, the transient solution can be considered as a parametric continuation, i.e., nonphysical solution caused by a lack of transition to turbulent flow modeling and transient turbulent modeling. Steady state sets in around $\tau \simeq 0.5$.

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Periodic Vibration of Plates with Large Displacements

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Nomenclature

$[K_{1b}]$, $[K_{1p}]$	=	linear bending and stretching stiffness matrices
$[K_2]$, $[K_3]$, and $[K_4]$	=	components of nonlinear stiffness matrix
$[M_b]$, $[M_p]$	=	bending and in-plane mass matrices
$[N]$	=	matrix of shape functions
$\{q\}$	=	generalized displacements
u and v	=	in-plane displacements
w	=	transverse displacement
α	=	damping parameter

Introduction

TO characterize the geometrically nonlinear dynamic behavior of plates, it is useful to define their periodic response to harmonic excitations in the frequency range of interest. There are several ways to carry out this task.[1] When finite element (FE) models are used, one often applies the harmonic balance method (HBM)[2] or the incremental HBM.[3] In these methods the number of nonlinear equations to solve simultaneously increases with the number of harmonics used and can be very large. Moreover, the model will be incorrect if the appropriate harmonics are not included in the Fourier series.

The time-domain shooting method[1,4] has two major advantages when compared with the HBM. First, the number of equations to solve is of the order of the original system. Second, it does not depend on an a priori assumption of the number of harmonics present in the motion's Fourier spectrum. Unlike time-domain integration methods applied on their own, like Newmark's method, the shooting technique provides a systematic procedure of calculating the periodic motions in a certain frequency range and converges to stable and unstable solutions. Moreover, the shooting method gives as a byproduct the monodromy matrix, the eigenvalues of which define the solutions' stability.[1]

In this work an algorithm based on the shooting and Newton methods is used to solve the FE equations of motion of isotropic plates. To validate it and to demonstrate that this algorithm has advantages when compared with other methods, results are compared with published ones.

Finite Element Equations of Motion; Shooting and Newton Methods

The hierarchical FE method used to model geometrical nonlinear vibrations of thin, elastic, isotropic plates is described in Ref. 2. The model is derived applying the d'Alembert's principle and the principle of virtual work. Considering stiffness proportional viscous damping,[5] a system of n equations of motion of the following form is derived:

$$\begin{bmatrix} [M_p] & 0 \\ 0 & [M_b] \end{bmatrix} \begin{Bmatrix} \ddot{q}_p \\ \ddot{q}_w \end{Bmatrix} + \alpha \begin{bmatrix} [K_{1p}] & 0 \\ 0 & [K_{1b}] \end{bmatrix} \begin{Bmatrix} \dot{q}_p \\ \dot{q}_w \end{Bmatrix} + \begin{bmatrix} [K_{1p}] & [K_2] \\ [K_3] & [K_{1b}] + [K_4] \end{bmatrix} \begin{Bmatrix} q_p \\ q_w \end{Bmatrix} = \begin{Bmatrix} P_p \\ P_w \end{Bmatrix} \tag{1}$$

or

$$[M]\{\ddot{q}\} + \alpha[K]\{\dot{q}\} + [KNL]\{q\} = \{P\} \tag{2}$$

The subscripts p and b indicate if the vectors and matrices are caused by the in-plane or bending effects.

Only fixed boundary conditions will be investigated, and, because in this case the middle plane in-plane displacements are much smaller than the transverse displacement, the in-plane inertia and damping will be neglected. The excitation vector $\{P\}$ is periodic with excitation frequency ω.

To apply the shooting method, the system of n second-order differential equations of motion (1) is transformed into the following $2n$ system of first-order differential equations:

$$\begin{bmatrix} 0 & [M] \\ [M] & \alpha[K] \end{bmatrix} \begin{Bmatrix} \dot{y} \\ \dot{q} \end{Bmatrix} + \begin{bmatrix} -[M] & 0 \\ 0 & [KNL] \end{bmatrix} \begin{Bmatrix} y \\ q \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix} \tag{3}$$

The period can be normalized to unity, by means of transformation $\tau = t/T$, so that the integration time interval is [0, 1]. Therefore, the system of differential equations (3) becomes

$$\begin{bmatrix} 0 & [M] \\ [M] & \alpha[K] \end{bmatrix} \begin{Bmatrix} y'(\tau) \\ q'(\tau) \end{Bmatrix} = T \left(\begin{Bmatrix} 0 \\ P \end{Bmatrix} - \begin{bmatrix} -[M] & 0 \\ 0 & [KNL] \end{bmatrix} \begin{Bmatrix} y(\tau) \\ q(\tau) \end{Bmatrix} \right) \tag{4}$$

where the prime denotes differentiation with respect to τ.

By using a $2n$ phase-space vector $\{X(\tau)\} = \{y(\tau), q(\tau)\}$, one can write an initial value problem related to the boundary-value problem (4) as follows:

Received 8 February 2001; presented as Paper 2001-1312 at the AIAA/ASME/ASCE/AHS/ASC 42nd Structures, Structural Dynamics, and Materials Conference, Seattle, WA, 16–19 April 2001; revision received 22 July 2001; accepted for publication 28 August 2001.

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$$\{X'(\tau)\} = T[\bar{M}]^{-1}(\{\bar{P}\} - [\bar{K}]\{X(\tau)\})$$
$$\{X(0)\} = \{s\}, \qquad \{X\}, \{s\} \in R^{2n} \tag{5}$$

where $\{s\}$ is a vector of initial conditions at $\tau = 0$. The definition of matrices $[\bar{M}]$ and $[\bar{K}]$ and of the vector $\{\bar{P}\}$ is carried out by comparing Eqs. (4) and (5). The solution of Eq. (5) is equal to the vector of initial conditions $\{s\}$ at $\tau = 0$ and 1. In summary, one is seeking for $\{s\}$, such that the following residual vector is close to zero:

$$\{r(\{s\}, \omega)\} = \{s\} - \{X(\{s\}, \omega, 1)\} \tag{6}$$

The first step in the shooting/Newton algorithm is to choose an excitation frequency and an initial guess $\{s\}^0$. The initial conditions for the first two points on the resonance curve, which correspond to the first two excitation frequencies, are the solution of the linear problem. For the following points of the response curve, a secant predictor[2] is used. Then, $\{s\}$ is corrected by

$$\{s\}^{v+1} = \{s\}^v + \Delta\{s\}^v \tag{7}$$

until convergence is achieved. The corrections $\Delta\{s\}^v$ solve the linear system of equations

$$[J(s^v, \omega)]\Delta\{s\}^v = -\{r(\{s\}^v, \omega)\tag{8}$$

The matrix $[J]$ in the former equation is the Jacobian of $\{r(\{s\}, \omega)\}$ with respect to $\{s\}$, i.e.,

$$[J(\{s\}^v, \omega)] = [[I] - [W(\{s\}^v, \omega, 1)]] \tag{9}$$

Matrix $[W(s, \omega, 1)]$ is obtained by solving the following initial value problem:

$$[W'] = [A][W], \qquad [W(s, \lambda, 0)] = [I] \tag{10}$$

where matrix $[A]$ is defined as $[A(\tau, \{s\}, \lambda)] = \partial(T[\bar{M}]^{-1}(\{\bar{f}\} - [\bar{K}]\{X(\tau)\}))/\partial\{X\}$. The system of equations of motion (5) and the system of equations (10) are simultaneously integrated to calculate $[W(\{s\}, \omega, 1)]$ and $\{X(\{s\}, \omega, 1)\}$. Then, the system of Eqs. (8) is solved, and $\{s\}$ is updated. When $\Delta\{s\}$ is sufficiently low, convergence to the periodic solution has been achieved, and one can proceed to the following point on the curve, by increasing ω. It is important to stress that for any periodic motion there are only $2n$ unknowns and $2n$ equations, independent of the number of harmonics present in the solutions Fourier spectrum.

The monodromy matrix is equal to $[W(\{s\}, \lambda, 1)]$ (Ref. 1). If an eigenvalue of this matrix, a Floquet multiplier, has a norm greater than one, then the solution is unstable.

Numerical Applications and Discussion

Two quadrangular, isotropic plates with fully clamped boundaries are analyzed. Table 1 compares the amplitudes $w_{\max}$ of displacement of an undamped steel plate, as a function of the frequency of excitation, calculated by the proposed approach with the ones from Refs. 2 and 6. In these references the HBM or similar methods were used. IDI stands for in-plane displacement and inertia. The plate material is steel, its width is $a = 500$ mm, and its thickness is $h = 2.0833$ mm. An uniform harmonic distributed force was applied. The hierarchical finite element method (HFEM) model uses three out-of-plane and six in-plane shape functions, as the one applied in conjunction with the HBM in Ref. 2. Figure 1 displays the plate's frequency response in the former conditions, as well as the displacement of the middle point of the plate ($x = 0$, $y = 0$) along one cycle and the Fourier spectrum for a particular frequency of excitation. W_i stands for the amplitude and ω_i for the frequency of the ith harmonic. At some excitation frequencies higher harmonics were found in the motion. This explains in part the different values in Table 1. Several

Table 1 Frequency ratio $\omega/\omega_{\ell 1}$ vs amplitude of vibration for a plate subjected to a harmonic distributed force $P_0 = 0.2$[a]

$w_{\max}/h$	FEM without IDI[6]	HFEM and shooting method	HFEM and HBM[2]	
			$w_{\max}/h$	$\omega/\omega_{\ell 1}$
±0.2	0.1180	0.2707	+0.2001	0.2442
	1.4195	1.4387	−0.2005	1.4399
±0.6	0.8905	0.8950	+0.5992	0.8962
	1.2083	1.2114	−0.5997	1.2114
±1	1.0700	1.0777	+1.000	1.0800
	1.2429	——	−1.001	1.2491

[a] From Ref. 6: $P_0 = cP_d/\rho h^2\omega_{\ell 1}^2$, $c = \iint \phi \, dx \, dy / \iint \phi^2 \, dx \, dy$, ϕ is normalized mode shape, and P_d is amplitude of external force (N/m^2).

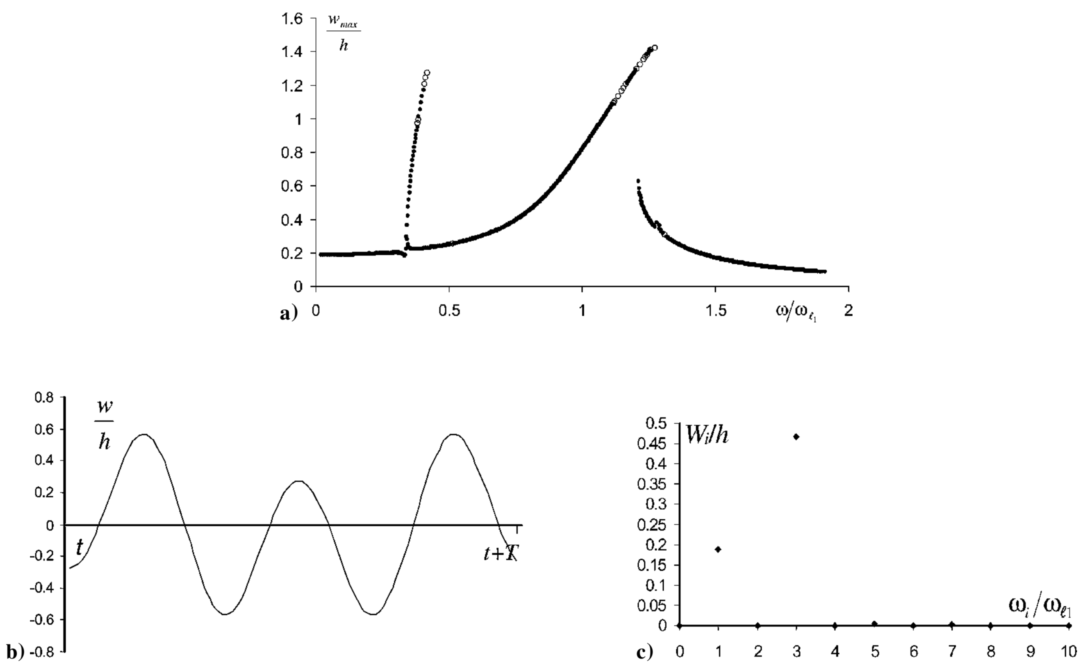

Fig. 1 Plate's frequency response: a) transverse displacement at $x = 0$, $y = 0$ in function of frequency with •, stable solutions and ○, unstable solutions; b) and c) displacement along one cycle and Fourier series at excitation frequency equal to 0.350 ω_{ℓ_1}, upper branch.

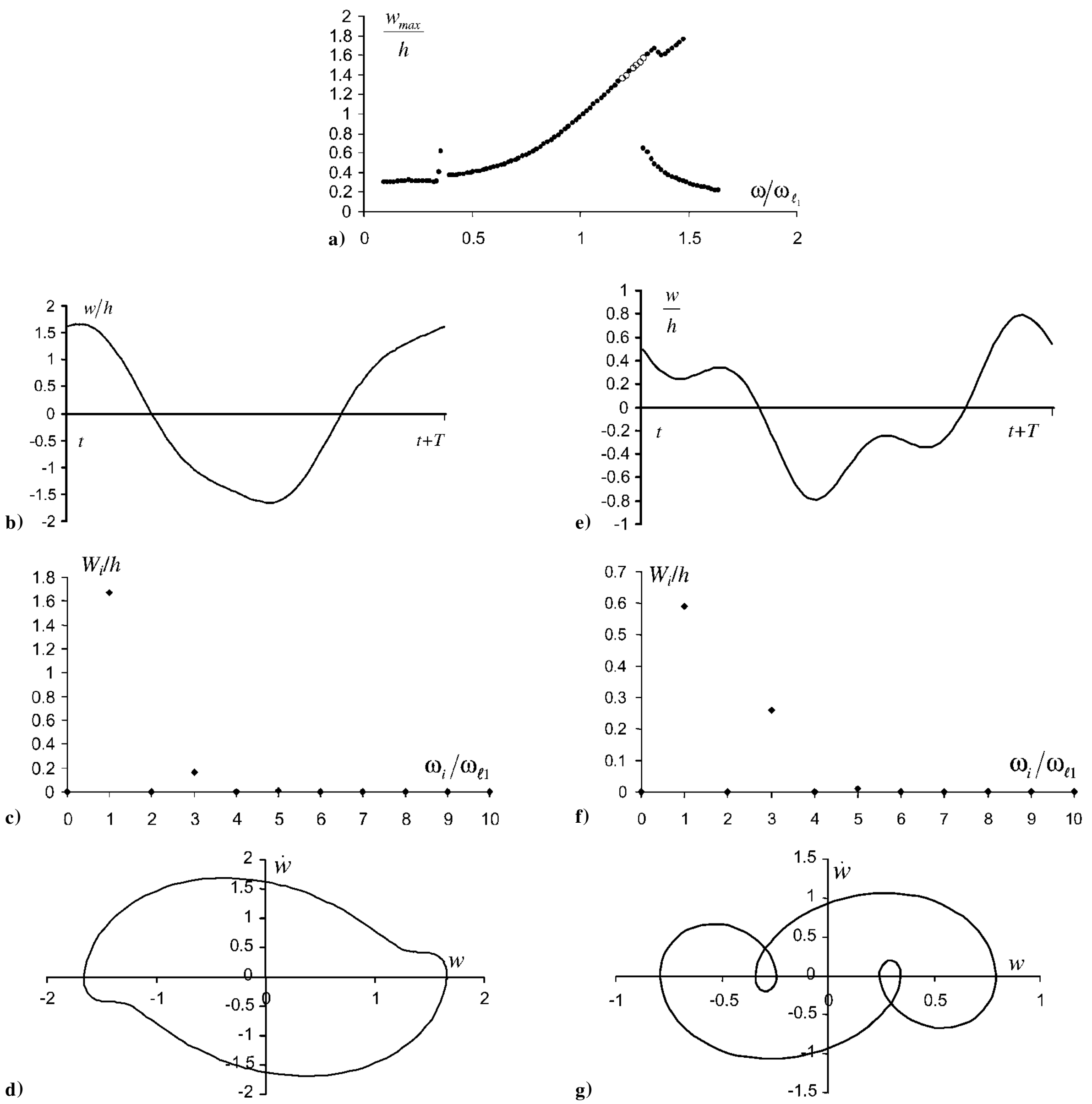

Fig. 2 Response of a fully clamped aluminum plate to a distributed harmonic excitation of amplitude 200 N/m^2: a) transverse displacement at $x = 0$, $y = 0$ with •, stable solutions and ○, unstable solutions. Displacement along one cycle, Fourier series, and phase space at excitation frequency equal to $1.3499\omega_{\ell_1}$; b), c), and d) $x = 0$ and $y = 0$; e), f), and g) $x = 3a/8$ and $y = 0$.

stability losses occurred; some when one Floquet multiplier crossed the unit circle through +1 and the others when a pair of complex conjugate Floquet multipliers crossed the unit circle.

In Fig. 2 the response of a fully clamped aluminum plate to a distributed harmonic excitation of amplitude 200 N/m^2 is represented. The plate's width is 300 mm, and the thickness is 1 mm. The value of α is 10^{-5}. Two losses of stability were found. The first occurs when a Floquet multiplier crosses the unit circle through +1 and the second when a pair of complex conjugate Floquet multipliers crosses the unit circle. As the phase planes and Fourier spectra show, at some excitation frequencies higher harmonics have a large influence in the response. Coupling between different vibration modes was also found.

Similar examples that demonstrate the advantages of the proposed methods occurred in plates with other boundary conditions[7] and in beams. For instance, in a clamped-clamped beam and at certain excitation frequencies the seventh harmonic was found to be quite important, whereas the third and the fifth were rather small. Thus, the seventh harmonic would probably be erroneously neglected was one using the HBM. Generally, it was found that higher damping causes the solutions to be more stable and decreases the importance of higher harmonics. Finally, all of the computations were carried out within reasonable computational time on a personal computer.

Conclusions

It was demonstrated that the shooting method and the finite element method allow one to investigate accurately geometrically nonlinear, periodic vibrations of plates. The fact that an a priori assumption of which and how many harmonics are present in the motion is not necessary constitutes a great advantage of the shooting method, particularly when compared with the harmonic balance method.

Acknowledgment

The support of this work by the Portuguese Science and Technology Foundation under Project POCTI/1999/EME/32641, FEDER, is gratefully acknowledged.

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